A THEOREM ON SEMI-CONTINUOUS SET-VALUED FUNCTIONS

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1. **Introduction.** The purpose of this paper is to prove a rather curious result about semi-continuous set-valued functions (Theorem 1.1), and to derive some applications to open and closed point-valued functions.

Recall that if X and Y are topological spaces, and if 2^Y denotes the space of non-empty subsets of Y, then a function $\phi: X \to 2^Y$ is called *lower* (respectively *upper*) *semi-continuous* if

$$\{x \in X \mid \phi(x) \cap U \neq \emptyset\} \qquad \text{(respectively} \qquad \{x \in X \mid \phi(x) \subset U\})$$

is open in X for every open $U \subset Y$. An important example is the case where ϕ is of the form $\phi = u^{-1}$ for some function u from Y onto X; in this case, ϕ is lower (respectively upper) semi-continuous if and only if u is open (respectively closed).

Theorem 1.1. *Let X be paracompact, Y a metric space, and $\phi: X \to 2^Y$ lower semi-continuous with each $\phi(x)$ complete. Then there exist $\psi: X \to 2^Y$ and $\theta: X \to 2^Y$ such that*

(a) *$\psi(x) \subset \theta(x) \subset \phi(x)$ for all $x \in X$,*
(b) *$\psi(x)$ and $\theta(x)$ are compact for all $x \in X$,*
(c) *ψ is lower semi-continuous,*
(d) *θ is upper semi-continuous.*

Corollary 1.2. *Let E be a metric space, F paracompact (Hausdorff will suffice in (b)), and $f: E \to F$ open and onto with $f^{-1}(y)$ complete for every $y \in F$. Then*

(a) *There exist subsets $E'' \subset E' \subset E$ such that $f(E'') = f(E') = F$, $f \mid E'$ is closed, $f \mid E''$ is open, and for each $y \in F$ the sets $(f \mid E')^{-1}(y)$ and $(f \mid E'')^{-1}(y)$ are compact.*

(b) *If $B \subset F$ is compact, then there exists a compact $A \subset E$ such that $f(A) = B$.*

(c) *If f is continuous, then F is metrizable.*

Observe that Corollary 1.2. (b) generalizes a result of Bourbaki [1, §2, Proposition 18], where f is continuous, and E itself—rather than just the sets $f^{-1}(y)$—is required to be complete.

It should be noted that the roles of open and closed maps in Corollary 1.2 (a) cannot be reversed. For instance, the map $f: [0,3] \to [0,2]$, defined by $f(x) = x$ if $0 \leq x \leq 1$, $f(x) = 1$ if $1 \leq x \leq 2$, and $f(x) = x - 1$ if $2 \leq x \leq 3$, is continuous and closed, but there exists no $A \subset [0, 3]$ such that $f(A) = [0, 2]$ and $f \mid A$ is open.

The proof of Theorem 1.1 will be found in §§2 and 3; the lemma in §2 may

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hold some independent interest, and will also be used in [4]. The short proof of Corollary 1.2 is contained in §4, and §5 gives an example showing why we need completeness for our results.

2. A lemma.

LEMMA 2.1. *Let X be paracompact, Y a topological space, $\phi: X \to 2^Y$ lower semi-continuous, and $\{\sigma_n\}_{n=1}^{\infty}$ a sequence of continuous pseudometrics on Y. Then for each integer $n > 0$ there exists an index set A_n, a locally finite open covering $\{U_\alpha\}_{\alpha \in A_n}$ of X, elements $y_\alpha(x) \in \phi(x)$ whenever $x \in \bar{U}_\alpha$ with $\alpha \in A_n$, and a map $\pi_n : A_{n+1} \to A_n$ onto, such that the following conditions are satisfied for all n.*

(a) *If $\alpha \in A_n$ and $x, x' \in \bar{U}_\alpha$, then $\sigma_n(y_\alpha(x), y_\alpha(x')) < \frac{2}{3}$.*

(b) *If $\alpha \in A_n$, then $U_\alpha = \bigcup \{U_\beta \mid \beta \in \pi_n^{-1}(\alpha)\}$.*

(c) *If $\alpha \in A_n$, $\beta \in \pi_n^{-1}(\alpha)$, and $x \in \bar{U}_\beta$, then $\sigma_n(y_\beta(x), y_\alpha(x)) < 1$.*

Proof. By induction. For convenience we start with $n = 0$, after defining the pseudometric σ_0 by $\sigma_0(y, y') = 0$ for all $y, y' \in Y$. We can therefore begin by letting A_0 be a one-element set, say $\{\nu\}$, letting $U_\nu = X$, and for each $x \in X$ letting $y_\nu(x)$ be any element of $\phi(x)$.

Suppose that we have everything up to n, and let us construct it for $n + 1$. Without loss of generality, we may assume that $\sigma_{n+1} \geq \sigma_n$. Let $\alpha \in A_n$. For each $x \in \bar{U}_\alpha$, let

$$V_\alpha(x) = \{x' \in \bar{U}_\alpha \mid \sigma_{n+1}(y_\alpha(x), \phi(x')) < \tfrac{1}{3}\}. \tag{1}$$

Then each $V_\alpha(x)$ is open in $\bar{U}_\alpha$ by the lower semi-continuity of ϕ, and if we let

$$\mathcal{V}_\alpha = \{V_\alpha(x) \mid x \in \bar{U}_\alpha\},$$

then $\mathcal{V}_\alpha$ is a relatively open covering of $\bar{U}_\alpha$. Since $\bar{U}_\alpha$ is paracompact, $\mathcal{V}_\alpha$ has a locally finite, relatively open refinement $\{W_\beta\}_{\beta \in B(\alpha)}$, which in turn has a relatively open refinement $\{W'_\beta\}_{\beta \in B(\alpha)}$, such that $\bar{W}'_\beta \subset W_\beta$ for all $\beta \in B(\alpha)$.

We may assume that the family of index sets $\{B(\alpha)\}_{\alpha \in A_n}$ is disjoint, so we let

$$A_{n+1} = \bigcup_{\alpha \in A_n} B(\alpha),$$

and we define $\pi_n : A_{n+1} \to A_n$ by

$$\pi_n(\beta) = \alpha \quad \text{if} \quad \beta \in B(\alpha).$$

Note that $\pi_n^{-1}(\alpha) = B(\alpha)$ for all $\alpha \in A_n$.

Now let $\beta \in A_{n+1}$. We must define the open set U_β, and the points $y_\beta(x)$ for $x \in \bar{U}_\beta$. Denote $\pi_n(\beta)$ by α. Let $U_\beta = W'_\beta \cap U_\alpha$. Pick an $x_\beta \in \bar{U}_\alpha$ such that $\bar{U}_\beta \subset V_\alpha(x_\beta)$, and then for each $x \in \bar{U}_\beta$ use (1) to pick $y_\beta(x) \in \phi(x)$ such that

$$\sigma_{n+1}(y_\beta(x), y_\alpha(x_\beta)) < \tfrac{1}{3}. \tag{2}$$

Let us verify our requirements. Since $\{U\beta \mid \beta \in \pi_n^{-1}(\alpha)\}$ is clearly a locally finite (with respect to X) open covering of U_α for all $\alpha \in A_n$, it follows, by our inductive assumption on $\{U_\alpha\}_{\alpha \varepsilon A_n}$, that $\{U_\beta\}_{\beta \varepsilon A_{n+1}}$ is a locally finite open covering of X. It remains to check conditions (a) $-$ (c). The validity of (b) is clear, and that of (a) (with n replaced by $n + 1$) follows from (2). As for (c), note that if $\alpha \varepsilon A_n$, $\beta \varepsilon \pi^{-1}(\alpha)$, and $x \varepsilon \bar{U}_\beta$, then $\sigma_n(y_\beta(x), y_\alpha(x)) \leq \sigma_n(y_\beta(x), y_\alpha(x_\beta)) + \sigma_n(y_\alpha(x_\beta), y_\alpha(x)) < \frac{1}{3} + \frac{2}{3} = 1$. This completes the proof.

3. **Proof of Theorem** 1.1. Begin by applying Lemma 2.1 with $\sigma_n = 2^n\rho$, where ρ is the metric on Y, and let A_n, π_n, U_α, and $y_\alpha(x)$ be as in Lemma 2.1. Now let

$$A = \{\alpha = \{\alpha_n\}_{n=1}^{\infty} \mid \alpha_n \varepsilon A_n \quad \text{and} \quad \pi_n(\alpha_{n+1}) = \alpha_n \quad \text{for all} \quad n\},$$

and for each $x \varepsilon X$, let

$$A(x) = \{\alpha \varepsilon A \mid x \varepsilon U_{\alpha_n} \quad \text{for all} \quad n\},$$
$$\bar{A}(x) = \{\alpha \varepsilon A \mid x \varepsilon \bar{U}_{\alpha_n} \quad \text{for all} \quad n\}.$$

Note first that if $x \varepsilon X$ and α is in $A(x)$ or $\bar{A}(x)$, then $y_{\alpha_n}(x)$ is defined for all n, and $\{y_{\alpha_n}(x)\}_{n=1}^{\infty}$ is a Cauchy sequence (by 2.1 (c)). Since $\phi(x)$ is complete, this sequence must have a limit, which we denote by $y_\alpha(x)$. We now let

$$\psi_0(x) = \{y_\alpha(x) \mid \alpha \varepsilon A(x)\},$$
$$\theta_0(x) = \{y_\alpha(x) \mid \alpha \varepsilon \bar{A}(x)\},$$

and we let

$$\psi(x) = (\psi_0(x))^-,$$
$$\theta(x) = (\theta_0(x))^-.$$

for every $x \varepsilon X$.

Before proving that the above ψ and θ satisfy our conditions, let us make some observations.

(1) If $\alpha \varepsilon \bar{A}(x)$, then $\rho(y_{\alpha_n}(x), y_\alpha(x)) < 2^{-(n-1)}$. (By 2.1 (c)).

(2) If $\alpha \varepsilon \bar{A}(x)$, $\alpha' \varepsilon \bar{A}(x')$, and $\alpha_n = \alpha'_n$, then $\rho(y_\alpha(x), y_{\alpha'}(x')) < 2^{-(n-3)}$. (This follows from (1) and 2.1 (a)).

(3) If $\beta \varepsilon A_n$ and $x \varepsilon U_\beta$, then there exists an $\alpha \varepsilon A(x)$ such that $\alpha_n = \beta$. (By 2.1 (b)).

(4) If $\beta \varepsilon A_n$ and $x \varepsilon \bar{U}_\beta$, then there exists an $\alpha \varepsilon \bar{A}(x)$ such that $\alpha_n = \beta$. (This also follows from 2.1 (b) which, together with local finiteness, implies that $\bigcup\{\bar{U}_\delta \mid \delta \varepsilon \theta_n^{-1}(\gamma)\} = \bar{U}_\gamma$).

It follows from (3) that $A(x)$ is never empty, and hence neither are $\psi(x)$ or $\theta(x)$. It remains to check requirements (a) $-$ (d).

(a) This is obvious from the definitions.

(b) It suffices to observe that, by (1) and local finiteness, $\theta_0(x)$—and hence certainly $\psi_0(x)$—is totally bounded for all x.

(c) It is clearly sufficient to show that ψ_0 is lower semi-continuous. So given $x \varepsilon X$, $y \varepsilon \psi_0(x)$, and $\epsilon > 0$, we must find a neighborhood W of x such that

$\psi_0(x')$ intersects $S_\epsilon(y)$ for all $x' \varepsilon W$ (where $S_\epsilon(y)$ denotes the open ϵ-sphere about y). Note first that $y = y_\alpha(x)$ for some $\alpha \varepsilon \mathrm{A}(x)$. Now pick an n such that $2^{-(n-3)} < \epsilon$, and let $W = U_{\alpha_n}$. If now $x' \varepsilon W$, then by (3) there exists an $\alpha' \varepsilon \mathrm{A}(x')$ such that $\alpha'_n = \alpha_n$. But then $y_{\alpha'}(x') \varepsilon \psi_0(x')$, and $\rho(y_{\alpha'}(x'), y_\alpha(x)) < \epsilon$ by (2), which is what we needed.

(d) We must show that if $x \varepsilon X$ and V is a neighborhood of $\theta(x)$ in Y, then there exists a neighborhood W of x in X such that $\theta(x') \subset V$ for all $x' \varepsilon W$. Now since $\theta(x)$ is compact, we can find $\epsilon > 0$ such that $S_\epsilon(\theta(x)) \subset V$ (where $S_\epsilon(E)$ denotes $\bigcup_{y \varepsilon E} S_\epsilon(y)$). It therefore suffices to pick W such that $\theta_0(x') \subset S_{\frac{1}{2}\epsilon}(\theta_0(x))$ for every $x' \varepsilon W$. To do this pick an n such that $2^{-(n-3)} < \frac{1}{2}\epsilon$, and let

$$W = X - \bigcup \{ \bar{U}_\beta \mid \beta \varepsilon \mathrm{A}(n), x \not\varepsilon \bar{U}_\beta \}.$$

Suppose that $x' \varepsilon W$ and $y \varepsilon \theta_0(x')$. Now $y = y_{\alpha'}(x')$ for some $\alpha' \varepsilon \bar{\mathrm{A}}(x')$. Then $x' \varepsilon \bar{U}_{\alpha'_n}$, hence $x \varepsilon \bar{U}_{\alpha'_n}$, and thus by (4) there exists an $\alpha \varepsilon \bar{\mathrm{A}}(x)$ such that $\alpha_n = \alpha'_n$. But then $y_\alpha(x) \varepsilon \theta_0(x)$, and $\rho(y_\alpha(x), y_{\alpha'}(x')) < \frac{1}{2}\epsilon$ by (2), which is all we had to show. This completes the proof of the theorem.

3. **Proof of Corollary** 1.2. (a) Define $\phi\colon F \to 2^E$ by $\phi(y) = u^{-1}(y)$. By the remark preceding Theorem 1.1, this ϕ is lower semi-continuous, and thus satisfies all the requirements of Theorem 1.1. With ψ and θ as in Theorem 1.1, let $E' = \bigcup_{y \varepsilon F} \theta(y)$ and $E'' = \bigcup_{y \varepsilon F} \psi(y)$. Again by the remark preceding Theorem 1.1, E' and E'' satisfy all our requirements.

(b) Applying (a) to the function $g = f \mid f^{-1}(B)$, we obtain a set $A \subset f^{-1}(B)$ such that $g(A) = B$, $g \mid A$ is closed, and $(g \mid A)^{-1}(y)$ is compact for every y in the compact set B. But this implies (see, for instance, [2, Theorem 1]) that A is compact.

(c) We need only apply to $f \mid E'$ (where E' is as in (a)) the theorem (see S. Hanai and K. Morita [3, Theorem 1] or A. H. Stone [5, Theorem 1]) which asserts that the continuous, closed image of a metrizable space is metrizable if (and only if) inverse images of points have compact boundaries. This comletes the proof.

4. **An example.** In this section we construct an exampleto show that 1.2(b) (and hence 1.2(a) and 1.1) can become false if one fails to assume completeness.

Example 4.1. Let π be the natural projection from the unit square S onto the unit interval F. Let E be a space obtained from S by removing from each vertical interval $\pi^{-1}(y)$ a point x_y, and let $f = \pi \mid E$. Then f is open (and continuous), but the sets $f^{-1}(y)$ are not complete. Let us show that the points x_y can be picked so that there exists no compact $A \subset E$ with $f(A) = F$.

Let $\mathfrak{A} = \{A \subset S \mid A \text{ is compact}, \pi(A) = F\}$. Then card $\mathfrak{A} = 2^{\aleph_0}$ (since S has a countable base), and hence there exists a function μ from F onto $\mathfrak{A}$. For every $y \varepsilon F$ we now let x_y be any element of $\mu(y) \cap \pi^{-1}(y)$. This works, for if $A \subset S$ is compact with $\pi(A) = F$, then $A = \mu(y)$ for some $y \varepsilon F$, whence $x_y \varepsilon A$, and hence $A \not\subset E$.

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